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Linearizability conditions for Lotka–Volterra planar complex cubic systems

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Abstract
In this paper, we investigate the linearizability problem for the two-dimensional planar complex system $\dot{x} = x(1 - a_{10}x - a_{01}y - a_{20}x^2 - a_{11}xy - a_{02}y^2)$, $\dot{y} = -y(1 - b_{10}x - b_{01}y - b_{20}x^2 - b_{11}xy - b_{02}y^2)$. The necessary and sufficient conditions for the linearizability of this system are found. From them the conditions for isochronicity of the corresponding real system can be derived.

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1. Introduction and statement of the results

Consider a planar analytic differential system in the form of a linear center perturbed by higher-order terms, that is,

$$\dot{u} = -v + U(u, v), \qquad \dot{v} = u + V(u, v), \tag{1}$$

where U and V are real analytic functions whose series expansions in a neighborhood of the origin start in at least second-order terms. Taking polar coordinates we can see that near the origin either all non-stationary trajectories of (1) are ovals (in which case the origin is a center) or they are spirals (in which case the origin is a focus). By the Poincaré–Lyapunov system, (1) has a center at the origin if and only if there is a first integral:

$$\Phi(u, v) = u^2 + v^2 + \sum_{k+j=3}^{\infty} \phi_{kj} x^k y^j,$$

where the series converges in a neighborhood of the origin. The problem of distinguishing between a center and a focus at the origin of system (1) is called the *center problem*.

If the origin is a center, then the problem that arises is to determine when the period of the solutions near the origin is constant. A center with such a property is called an *isochronous center*. The research of the isochronous center phenomena was started in 1673 when Huygens studied the cycloidal pendulum; see [16]. The isochronous center theorem of Poincaré and Lyapunov says that the center of (1) is isochronous if and only if it is linearizable. Hence, the isochronicity problem is equivalent to the linearizability problem. The history of the study of the isochronous center is interesting also in the sense that this result has been rediscovered several times. It was rediscovered by Gregor [24] in 1958, Lukashevich [31] in 1965, Brickman–Thomas [2] in 1977 and Villarini [39] in 1992; see, for instance, [20, 21, 42] and references therein. There are only few families of polynomial differential systems in which a complete classification of the isochronous centers is known; see, for instance, [8, 9, 27, 30, 31, 35, 37].

In 1964, Loud [30] classified isochronous centers of system (1) with U and V being the homogeneous polynomials of degree 2, and in 1969, Pleshkan [35] found all isochronous centers in the family (1), where U and V are homogeneous polynomials of degree 3. However, the classifications of isochronous centers in the form of a linear center perturbed by homogeneous polynomials of the fourth and fifth degrees turned out to be much more difficult. In [8, 9], isochronous centers for time-reversible systems (1) in the case of a homogeneous perturbation of the fourth and fifth degrees were found (by definition, system (1) is time reversible if it is invariant under reflection with respect to a line passing through the origin and a change in the direction of time). More recently in [10], all the isochronous centers for time-reversible systems (1) in the case of a homogeneous perturbation of degree 4 have been obtained. Moreover, the complete classification of the isochronous centers for a linear center perturbed by fifth degree homogeneous polynomials has been given in [36].

It is known from the time of Lyapunov [26] that computation of normal forms and first integrals of system (1) can be considerably simplified if we introduce a complex structure on the phase plane (u, v) by setting $x = u + \mathrm{i}v$. Then we obtain from system (1) the equation

$$\dot{x} = R(x, \bar{x}).$$

Adjoining to the later equation its complex conjugate, we have the system

$$\dot{x} = R(x, \bar{x}), \qquad \dot{\bar{x}} = \bar{R}(x, \bar{x}).$$

Let us consider $\bar{x}$ as a new variable y and $\bar{R}$ as a new function. Then, in the case when U and V are polynomials of degree n, from the later system we obtain, after the change of time $\mathrm{i}dt = d\tau$ and rewriting t instead of τ, a system of two complex differential equations of the form

$$\begin{aligned} \dot{x} &= x - \sum_{p+q=1}^{n-1} a_{p,q} x^{p+1} y^q = P(x, y), \\ \dot{y} &= -y + \sum_{p+q=1}^{n-1} b_{p,q} x^q y^{p+1} = Q(x, y). \end{aligned} \tag{2}$$

Here $p \geqslant -1$ and $q \geqslant 0$, and we denote the coefficients of system (2) by $(A, B) = (a_{1,0}, a_{0,1}, \ldots, a_{-1,n}, b_{1,0}, b_{0,1}, \ldots, b_{n,-1})$. It is said that system (2) has a *center* at the origin if it admits a first integral of the form

$$\Psi(x, y) = xy + \sum_{l+j=3}^{\infty} v_{l,j} x^l y^j. \tag{3}$$

The linearizability problem for system (2) is the problem of deciding whether the system can be transformed into the linear system, $\dot{z}_1 = z_1$ and $\dot{z}_2 = -z_2$, by means of the formal change of the phase variables:

$$z_1 = x + \sum_{m+j=2}^{\infty} c_{m-1,j}(A,B)x^m y^j, \qquad z_2 = y + \sum_{m+j=2}^{\infty} d_{m,j-1}(A,B)x^m y^j. \tag{4}$$

If such a transformation exists, we say that the system is *linearizable*, i.e., there is a *linearizable center* at the origin. Moreover, it is well known that if there exists a formal series (4) linearizing (2), then there exists a series which converges in the neighborhood of the origin; see [1]. The linearizability problem for the complex system (2) is a generalization of the linearizability problem (isochronicity) for the real system (1) in the sense that if we know all linearizable centers within a given family (2) then going back to the real coordinates (u, v) we obtain all linearizable (isochronous) centers of family (1).

The center and isochronicity problems for the so-called cubic system, that is, for the system

$$\dot u = -v + U_2(u,v) + U_3(u,v), \qquad \dot v = u + V_2(u,v) + V_3(u,v), \tag{5}$$

where U_2, V_2 and U_3, V_3 are homogeneous polynomials of degrees two and three, respectively, have been intensively studied during last few decades (it appears that there are tens or maybe even hundreds of papers where the center and isochronicity problems for particular subfamilies of (5) have been discussed (see, e.g., [3–5, 7, 12, 15, 20, 23, 32, 37, 38] and references therein). In this paper, we study the linearizability problem for the following two-dimensional planar complex system:

$$\begin{aligned} \dot x &= x(1 - a_{10}x - a_{01}y - a_{20}x^2 - a_{11}xy - a_{02}y^2), \\ \dot y &= -y(1 - b_{10}x - b_{01}y - b_{20}x^2 - b_{11}xy - b_{02}y^2). \end{aligned} \tag{6}$$

This ten-parameter cubic system contains the family studied in [15] and other families studied in [37, 38]. To the best of our knowledge it is the largest subfamily of the cubic system that has been studied so far. The family has two complex invariant lines passing through the origin, which is why we call it the complex Lotka–Volterra system. The differential equations modeling the interaction of two species have been studied extensively by real systems of the form

$$\dot x = xF(x,y), \qquad \dot y = yG(x,y),$$

being also known as *Kolmogorov systems*; see [28, 29, 41]. In that case, attention is restricted to the behavior of orbits in the 'realistic quadrant' $\{(x,y) \in \mathbb{R}^2 | x > 0, y > 0\}$. In the classical Lotka–Volterra model, F and G are linear, and it is well known that there are no limit cycles. If F and G are quadratic, of particular significance in applications is the existence of limit cycles and the number of them that can arise. These types of problems go back to Coleman [13] who poses a question: whether a predator–prey system can have two or more ecologically stable limit cycles. A first step to approach the problem of the existence and number of limit cycles is to solve the center problem for these real systems.

We also mention that the following question regarding a particular subfamily of system (6) has been raised in [32, p 231].

Problem 1. *Let (A) be the family of all systems (5) with two invariant lines $u \pm iv = 0$ and two other (real or complex) lines. What are the isochronous systems and the Darboux linearizable systems inside the family?*

Since the family (6) contains all systems (5) with two invariant lines $u \pm iv = 0$, it contains also the family (A). Thus, we find all isochronous centers not only inside (A), but within a much richer family. We discuss, in more detail, how our results are connected to the problem of isochronicity in the final section.

2. Preliminaries

In this section, we briefly describe the general approach used to study the linearizability problem for the polynomial system (2). The first step is the calculation of the so-called linearizability quantities, which are polynomials of the coefficients $a_{i,j}$ and $b_{i,j}$ of system (2). Taking derivatives with respect to t in both parts of each of the equalities in (4) and equating coefficients of the terms $x^{q_1+1}y^{q_2}$ and $x^{q_1}y^{q_2+1}$ we obtain the recurrence formulae,

$$(q_1 - q_2)c_{q_1,q_2} = \sum_{s_1+s_2=0}^{q_1+q_2-1} \left[(s_1+1)c_{s_1,s_2}a_{q_1-s_1,q_2-s_2} - s_2 c_{s_1,s_2}b_{q_1-s_1,q_2-s_2}\right], \tag{7}$$

$$(q_1 - q_2)d_{q_1,q_2} = \sum_{s_1+s_2=0}^{q_1+q_2-1} \left[s_1 d_{s_1,s_2}a_{q_1-s_1,q_2-s_2} - (s_2+1)d_{s_1,s_2}b_{q_1-s1,q_2-s2}\right], \tag{8}$$

where $s_1, s_2 \geqslant -1, q_1, q_2 \geqslant -1, q_1 + q_2 \geqslant 0, c_{1,-1} = c_{-1,1} = d_{1,-1} = d_{-1,1} = 0, c_{0,0} = d_{0,0} = 1$, and we set $a_{p,q} = b_{p,q} = 0$ if $p + q < 1$. Hence, we compute c_{q_1,q_2} and d_{q_1,q_2} of the formal change of variables (4) step by step using formulae (7) and (8). In the case $q_1 = q_2 = q$, the coefficients $c_{q,q}$ and $d_{q,q}$ can be chosen arbitrary (we set $c_{q,q} = d_{q,q} = 0$). The system is linearizable if and only if the quantities on the right-hand side of (7) and (8) are equal to zero for all $q_1 = q_2 = q \in \mathbb{N}$. In the case $q_1 = q_2 = q$, we denote the polynomials on the right-hand side of (7) by i_q and on the right-hand side of (8) by $-j_q$, calling them the *qth linearizability quantities*. Hence, system (2) with the given coefficient (A, B) is linearizable if, and only if, $i_q(A, B) = j_q(A, B) = 0$ for all $q \in \mathbb{N}$.

In the space of the parameters of a given family of systems (6) the set of all linearizable systems is an affine variety V of the ideal $\langle i_1, j_1, i_2, j_2, \ldots\rangle$. We recall that the variety of a given polynomial ideal $F = \langle f_1, f_2, \ldots, f_s\rangle$ is the set of common zeros of polynomials $f_1, f_2, \ldots, f_s$, and it is denoted by $V(F)$. Denote by I_k the ideal generated by the first k pairs of the linearizability quantities,

$$I_k = \langle i_1, j_1, \ldots, i_k, j_k\rangle. \tag{9}$$

By the Hilbert basis theorem, there exists $N \in \mathbb{N}$ such that V is equal to the variety of the ideal I_N, $V = V(I_N)$. However, the theorem does not give a constructive procedure to find N. In practice, N_0 is taken such that

$$V(I_{N_0}) = V(I_{N_0+1}). \tag{10}$$

Subsequently, the minimal associated primes of the ideal $I_{N_0} = \langle i_1, j_1, \ldots, i_{N_0}, j_{N_0}\rangle$, which define the irreducible decomposition of the variety $V(I_{N_0})$ of the ideal I_{N_0}, are computed, and for each component one tries to find the linearizing transformation (4). The most powerful method to find linearizing transformations is the so-called *Darboux linearization*, see [11, 12, 22, 33]. A smooth function, $F(x, y)$, satisfying

$$\frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial y}\dot{y} = KF, \tag{11}$$

is called a *Darboux factor* of system (6), and the polynomial $K(x, y)$ is called the *cofactor*.

The following theorem allows us to construct linearizing substitutions if sufficiently many Darboux factors are known.

Theorem 1 ([15]). *Assume that the coordinate axes are trajectories of system (2) and the system has Darboux factors $F_i(x, y)$ satisfying $F_i(0, 0) = 1$ with the cofactors $K_i(x, y)$, $i = 1, \ldots, s$. If*

$$(1-c)\frac{P(x,y)}{x} - c\frac{Q(x,y)}{y} + \sum_{i=1}^{s}\alpha_i K_i = 1 \tag{12}$$

for some $c, \alpha_1, \ldots, \alpha_s \in \mathbb{C}$, then the first equation of (2) can be linearized by the substitution $X = x^{1-c}y^{-c}\Psi^c f_1^{\alpha_1}\cdots f_s^{\alpha_s}$, and if

$$-c\frac{P(x,y)}{x} + (1-c)\frac{Q(x,y)}{y} + \sum_{i=1}^{s}\beta_i K_i = -1 \tag{13}$$

for some $c, \beta_1, \ldots, \beta_s \in \mathbb{C}$, then the second equation of (2) can be linearized by the substitution $Y = x^{-c}y^{1-c}\Psi^c f_1^{\beta_1}\cdots f_s^{\beta_s}$, where Ψ is a first integral of the form (3).

The theorem is a generalization of the theorems of [32, 33]. The difference between theorem 1 (and other generalizations, see also [22])) and the Darboux linearization theorems of [32, 33] is that they only permit invariant algebraic curves and exponential Darboux factors to construct linearizations, but in our generalization we allow any function with a cofactor. Thus, it allows us to work with some unknown functions as a first integral or an inverse integrating factor, but with known cofactors. For example, the first integral is a Darboux factor with the cofactor zero, and the inverse integrating factor is a Darboux factor with the cofactor being the divergence of the vector field.

If system (2) has a first integral (3) and a linearization of one of the equations of (2) is known, say the second equation is linearizable by $z_2 = Y(x, y)$, then the first one is linearizable by the substitution $z_1 = \Psi(x, y)/Y(x, y)$.

3. The linearizability conditions

In this section, we will find the conditions for linearizability of system (6). For this system using a straightforward modification of Mathematica code from [37], we have computed the first seven pairs of linearizability quantities. The polynomials are too long, so we do not give them here. The interested reader can easily compute them by using any available computer algebra system with the algorithms of [12, 37], for instance. Thus, we obtained the increasing chain of ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_7$ and the decreasing chain of varieties $\mathbf{V}(I_1) \supseteq \mathbf{V}(I_2) \supseteq \cdots \supseteq (I_7)$, where I_s are ideals (9). By the radical membership test (see, e.g., [14, chapter 4]), a polynomial f vanishes on the variety of the ideal $J \subset \mathbb{C}[x_1, \ldots, x_n]$ if and only if a Groebner basis of the ideal $\langle 1 - wf, J\rangle \subset \mathbb{C}[w, x_1, \ldots, x_n]$ is equal to one. Using the test we checked that both i_7 and j_7 vanish on the variety $\mathbf{V}(I_6)$, that is, (10) holds with $N_0 = 6$. Thus, we guess that the chain of the varieties $\mathbf{V}(I_1) \supseteq \mathbf{V}(I_2) \supseteq \cdots$ stabilizes on $\mathbf{V}(I_6)$, that is, $\mathbf{V}(I_6) = \mathbf{V}(\langle i_1, j_1, i_2, j_2, \ldots\rangle)$, and now we have to prove that this is indeed the case.

Note that for (6) $i_1 = 3a_{01}a_{10} + 3a_{11} + 3a_{01}b_{10}$ and $j_1 = -3a_{01}b_{10} - 3b_{01}b_{10} - 3b_{11}$, so from now on we assume that in (6) $a_{11} = -a_{01}a_{10} - b_{10}a_{01}$ and $b_{11} = -b_{01}b_{10} - b_{10}a_{01}$. To find necessary conditions for the linearizability of system (6) it is sufficient to find the irreducible decomposition of the variety $V(I)$ of the ideal $I = \langle i_1, j_1, \ldots, i_6, j_6\rangle$. To do so, we used the routine *minAssGTZ* [34] of Singular [25] which finds the minimal associated primes of a polynomial ideal by means of the Gianni–Trager–Zacharias method [19]. Note that if for system (6) $a_{01} \neq 0$ and $b_{10} \neq 0$, then by a linear transformation we can set in (7)

$a_{01} = b_{10} = 1$. Using this observation in order to simplify calculations, we split system (6) into three systems considering separately the cases:

$$(\alpha) a_{01} = b_{10} = 1, \qquad (\beta) a_{01} = 1, b_{10} = 0, \qquad (\gamma) a_{01} = b_{10} = 0.$$

For the case (α) we have obtained the necessary and sufficient conditions to have a linearizable center at the origin presented in theorem 2. The other remaining cases are studied in theorems 3 and 4. Note that if we apply to the conditions of theorem 3, which give the conditions for the case (β), the involution $a_{ij} \leftrightarrow b_{ji}$ then we obtain the integrability conditions of system (6) for the case $a_{01} = 0, b_{10} = 1$. Thus, theorems 2–4 provide the complete solution to the linearizable center problem of system (6). In the proof of theorem 2, we briefly describe an approach to find the necessary linearizability conditions, and then we show that the obtained conditions are also the sufficient conditions for a linearizable center. For cases (1), (2), (4), (5) of theorem 2, cases (1–5) of theorem 3 and all cases of theorem 4. the sufficiency of the conditions is established by applying theorem 1, that is, looking for Darboux factors of each family in order to construct a linearizing transformation of the form (4); however, to treat the other cases we have developed an approach based on another idea (which has some common features with the approach of [17]).

Theorem 2. *System (6) with $a_{01} = b_{10} = 1$ is linearizable at the origin if and only if $a_{11} = -a_{10} - a_{01}$ and $b_{11} = -b_{01} - b_{10}$, and one of the following conditions holds:*

(1) $b_{01} - b_{20} - b_{02} + 1 = a_{02} + b_{02} = a_{20} + b_{20} = a_{10} + b_{20} + b_{02} + 1 = 0,$
(2) $b_{01} - 3b_{20} + 1 = a_{02} = a_{20} - b_{20} = a_{10} - b_{20} + 1 = 2b_{20}^2 - 2b_{20} + b_{02} = 0,$
(3) $2a_{02} + b_{01} + 1 = a_{10} + 2b_{20} + 1 = 2b_{20}b_{02} - b_{01} + 2b_{02} - 1 = 2a_{20}b_{02} - b_{01} + 2b_{20} - 1$ $= b_{20}^2 + a_{20} + b_{20} = b_{01}b_{20} + b_{01} - b_{20} + 1 = b_{01}^2 + 4b_{02} - 1 = a_{20}b_{01} - a_{20} - 2b_{20} = 0,$
(4) $b_{20} = b_{01} - b_{02} + 1 = a_{02} = a_{20} - b_{02} = a_{10} - b_{02} + 1 = 0,$
(5) $b_{20} = b_{01} - b_{02} + 1 = a_{02} - b_{02} = a_{10} - 3b_{02} + 1 = 2b_{02}^2 + a_{20} - 2b_{02} = 0,$
(6) $b_{02} = b_{20} - 2 = b_{01} = 2a_{02} + 1 = a_{20} + 4 = a_{10} + 4 = 0,$
(7) $b_{02} + 4 = 2b_{20} + 1 = b_{01} + 4 = a_{02} - 2 = a_{20} = a_{10} = 0,$
(8) $4b_{02} + 3 = b_{20} - 6 = b_{01} - 2 = 8a_{02} + 3 = a_{20} - 12 = a_{10} + 4 = 0,$
(9) $b_{02} - 12 = 8b_{20} + 3 = b_{01} + 4 = a_{02} - 6 = 4a_{20} + 3 = a_{10} - 2 = 0.$

Proof. To obtain the necessary condition for the linearizable center of system (6) with $a_{01} = b_{10} = 1$ we look for the irreducible decomposition of the variety of the ideal $I = \langle i_1, j_1, \ldots, i_6, j_6 \rangle$, where $a_{01} = b_{10} = 1$. This is an extremely difficult computational problem which represents the most laborious part of our study. To perform the decomposition, we have used our computational approach, described in detail in [36].

Making use of the routine *minAssGTZ* of the computer algebra system Singular [25], we have found the irreducible decomposition of the variety of the ideal $I = \langle i_1, j_1, \ldots, i_6, j_6 \rangle$. The obtained decomposition consists of ten components. Then using the rational reconstruction algorithm [40], we go back to rational arithmetics. Performing the check similarly as in [36, p 5912], we see that one of the ten components is not a true one; the remaining nine, given in the statement of the theorem, give the correct decomposition. The usage of the modular arithmetics for the decomposition of varieties defined by coefficients of normal forms of system (2) goes back to the work by Edneral [18]. The difference of our approach is that we perform an additional check as described in [36].

We now show that under these conditions the system is linearizable. We can omit cases 5, 7 and 9 since they are dual to 2, 6 and 8, respectively, under the involution $a_{ij} \leftrightarrow b_{ji}$.

In order to find the Darboux factors we look for them in the form $F = \sum_{i+j=0}^{n} \alpha_{ij} x^i y^j$ with $K = \sum_{i+j=0}^{m-1} \beta_{ij} x^i y^j$ (m is the degree of the system, in our case $m = 2$; finding a bound for n is the Poincaré problem, still unresolved). Substituting this expression in (11) and equaling the coefficients of the same terms on the both sides of the equality, we obtain a system of polynomial equations in variables α_{ij}, β_{ij}. If a solution of the system exists, then solving the system we find a Darboux factor F (or a few such factors) of (2), which geometrically is the trajectory of (2) defined by $F(x, y) = 0$. Since F is a polynomial, it is called an algebraic invariant curve of (2).

For case 1, the corresponding system is written as

$$\dot{x} = x(1 + b_{20}x^2 + (b_{02} + b_{20} + 1)x - (b_{02} + b_{20})yx + b_{02}y^2 - y),$$
$$\dot{y} = -y(1 - b_{20}x^2 + (b_{02} + b_{20})yx - x - b_{02}y^2 - (b_{02} + b_{20} - 1)y).$$

Using the described approach we find that the system has the invariant lines:

$$\ell_1 = 1 + \tfrac{1}{2}\big(b_{02} + b_{20} - \sqrt{(b_{02} + b_{20} - 1)^2 + 4b_{02}} + 1\big)x + \tfrac{1}{2}\big(-b_{02} - b_{20} - \sqrt{(b_{02} + b_{20} - 1)^2 + 4b_{02}} + 1\big)y,$$

and

$$\ell_2 = 1 + \left(\frac{b_{02}}{2} + \frac{b_{20}}{2} + \frac{1}{2}\sqrt{(b_{02} + b_{20} - 1)^2 + 4b_{02}} + \frac{1}{2}\right)x + \frac{1}{2}\big(-b_{02} - b_{20} + \sqrt{(b_{02} + b_{20} - 1)^2 + 4b_{02}} + 1\big)y,$$

which yield the Darboux linearization

$$z_1 = x\ell_1^{\alpha}\ell_2^{\beta}, \qquad z_1 = y\ell_1^{\gamma}\ell_2^{\delta},$$

where

$$\alpha = \frac{b_{02} + b_{20} - \triangle + 1}{2\triangle}, \qquad \beta = \frac{b_{02} + b_{20} + \triangle + 1}{2\triangle},$$
$$\gamma = -\frac{b_{02} + b_{20} + \triangle - 1}{2\triangle}, \qquad \delta = \frac{b_{02} + b_{20} - \triangle - 1}{2\triangle},$$

and $\triangle = \sqrt{b_{02}{}^2 + 2(b_{20} + 1)b_{02} + (b_{20} - 1)^2}$.

In case 2, the system admits the invariant lines:

$$\ell_1 = 1 - b_{20}x, \qquad \ell_2 = 1 - \frac{2xyb_{20}^2}{b_{20} - 2} - 2yb_{20},$$
$$\ell_3 = 2x^2yb_{20}^2 + 4xyb_{20}^2 + 2yb_{20}^2 - 2xb_{20} - 4xyb_{20} - 3yb_{20} - 2b_{20} + x + y + 1,$$

and a Darboux linearization is given by

$$z_1 = x\ell_1^{\alpha_1}\ell_2^{\alpha_2}\ell_3^{\alpha_3}, \qquad z_2 = y\ell_1^{\alpha_4}\ell_2^{\alpha_5}\ell_3^{\alpha_6},$$

where

$$\alpha_1 = -\frac{b_{20}}{b_{20} + 1}, \qquad \alpha_2 = \frac{1}{b_{20} + 1}, \qquad \alpha_3 = -\alpha_2,$$
$$\alpha_4 = 2\alpha_2, \qquad \alpha_5 = 2\alpha_1, \qquad \alpha_6 = \frac{b_{20} - 1}{b_{20} + 1}.$$

The system of case 3 is written as

$$\begin{aligned} \dot{x} &= x\left(b_{20}(b_{20} + 1)x^2 + (2b_{20} + 1)x - 2b_{20}yx + \frac{b_{20}y^2}{b_{20} + 1} - y + 1\right) = P(x, y), \\ \dot{y} &= -y\left(-b_{20}x^2 + \frac{2b_{20}yx}{b_{20} + 1} - x - \frac{b_{20}y^2}{(b_{20} + 1)^2} - \frac{(b_{20} - 1)y}{b_{20} + 1} + 1\right) = Q(x, y). \end{aligned} \tag{14}$$

In addition to the coordinate axes it has the invariant line,

$$\ell_1 = b_{20}x - \frac{b_{20}y}{b_{20}+1} + 1.$$

However it is impossible to find a Darboux integral or an integrating factor using these lines. Although we are not able to find a closed form for a first integral (3) of system (14) we prove that such an integral exists. To this end, we make the substitution $z = y/x$. In the new coordinates (14) takes the form

$$\begin{aligned}\dot{x} &= x\big(x^2b_{20}^3 + 2x^2b_{20}^2 + 2xb_{20}^2 - 2x^2zb_{20}^2 + x^2b_{20} + x^2z^2b_{20} + 3xb_{20}\\ &\quad - 2x^2zb_{20} - xzb_{20} + b_{20} + x - xz + 1\big)\big/(b_{20}+1),\\ \dot{z} &= -z\big(x^2b_{20}^4 + 2x^2b_{20}^3 + 2xb_{20}^3 - 2x^2zb_{20}^3 + x^2b_{20}^2 + x^2z^2b_{20}^2 + 4xb_{20}^2 - 2x^2zb_{20}^2\\ &\quad - 2xzb_{20}^2 + 2b_{20}^2 + 2xb_{20} - 2xzb_{20} + 4b_{20} + 2\big)\big/(b_{20}+1)^2.\end{aligned} \tag{15}$$

We look for a first integral of (15) in the form

$$\psi(x,z) = x^2z\sum_{k=2}^{\infty} f_k(z)x^k, \tag{16}$$

where $f_k(z)$ are polynomials satisfying a first-order linear differential equation,

$$f_k' - \frac{k}{2z}f_k + a_1f_{k-1}' + a_2f_{k-2}' + \frac{b_1}{x}f_{k-1} + \frac{b_2}{z}f_{k-2} = 0,$$

where a_i, b_i for $i = 1, 2$ stand for some polynomials in z of degree i. Solving the equation, we obtain $f_2(z) = z$, $f_3(z) = z\big(-4b_{20} - \frac{4z}{b_{20}+1} - 4\big)$ and by induction we conclude that the other function f_k can be chosen in the form

$$f_k(z) = zp_{k-2}(z),$$

where $p_{k-2}(z)$ is a polynomial of degree $k-2$ in z. Then $\Psi(x,y) = \sqrt{\psi(x, y/x)}$ is a first integral of (14) of the form (3). (In more detail, the approach is described for case 6 of theorem 3 below.)

It is obvious that we can write system (2) with a first integral (3) in the form

$$\dot{x} = r\Psi_y, \qquad \dot{y} = -r\Psi_x, \tag{17}$$

for some analytic function $r(x,y)$ with $r(0,0) = 1$. Eliminating Ψ in (17) gives us $\dot{r} = \mathrm{div}(x,y)r$, which means that r is a Darboux function with the cofactor $\mathrm{div}(x,y)$, and therefore r is an inverse integrating factor. Equation (12),

$$(1-f)P(x,y)/x - fQ(x,y)/y + cK_1 + h\,\mathrm{div}(x,y) - 1,$$

where K_1 is the cofactor of ℓ_1, and P, Q are the right-hand sides of (14), has the solution

$$c = \frac{2b_{20}}{b_{20}+2}, \qquad f = \frac{1}{b_{20}+2}, \qquad h = -\frac{b_{20}}{b_{20}+2}.$$

Therefore, the linearization of (14) is given by

$$z_1 = \Psi^{\frac{1}{b_{20}+2}} r^{-\frac{b_{20}}{b_{20}+2}} x^{1-\frac{1}{b_{20}+2}} y^{-\frac{1}{b_{20}+2}} \ell_1^{\frac{2b_{20}}{b_{20}+2}}, \qquad z_2 = \Psi^{\frac{b_{20}+1}{b_{20}+2}} r^{\frac{b_{20}}{b_{20}+2}} x^{-\frac{b_{20}+1}{b_{20}+2}} y^{1-\frac{b_{20}+1}{b_{20}+2}} \ell_1^{-\frac{2b_{20}}{b_{20}+2}}.$$

For case 4, we found the invariant lines,

$$\ell_1 = 1 - b_{02}y, \quad \ell_2 = 1 - b_{02}x, \quad \ell_3 = b_{02}x + 2b_{02}xy + (b_{02}-1)y - x + b_{02} - 1,$$

which allow us to obtain the Darboux linearization,

$$z_1 = x\ell_1^{\alpha}\ell_2^{-b_{02}\alpha}\ell_3^{-\alpha}, \qquad z_2 = y\ell_1^{-b_{02}\alpha}\ell_2^{\alpha}\ell_3^{-\alpha},$$

where $\alpha = 1/(1+b_{02})$.

Case 6 is treated similarly to case 3. Performing the change, $z = y/x$, and looking for a first integral of the obtained system in the form

$$\psi(x, z) = x^2 z \sum_{k=2}^{\infty} f_k(z) x^k,$$

one can easily see that the functions $f_k(z)$ can be found recursively in the form

$$f_k(z) = z p_{k-1}(z),$$

where $p_{k-1}(z)$ is a polynomial of degree $k-1$ in z. Then $\Psi = \sqrt{\psi(x, y/x)}$ is a first integral of the form (3) and using (12) we find that a linearization is given by the substitution

$$z_1 = \frac{\Psi^{5/4} r^{1/4}}{x^{1/4} y^{5/4}}, \qquad z_2 = \frac{\Psi}{z_1},$$

where r is defined from (17) and therefore r is an inverse integrating factor.

Case 8: the corresponding system is written as

$$\begin{aligned} \dot{x} &= x\left(-12x^2 - 3yx + 4x + \frac{3y^2}{8} - y + 1\right) = P(x, y), \\ \dot{y} &= -y\left(-6x^2 + 3yx - x + \frac{3y^2}{4} - 2y + 1\right) = Q(x, y). \end{aligned} \tag{18}$$

We look for a linearization of the second equation of the system $Y = Y(x, y)$. The function Y should satisfy the equation

$$\frac{\partial Y}{\partial x} P + \frac{\partial Y}{\partial y} Q + Y = 0.$$

After the substitution $x = yz$, the latter equation is written as

$$\frac{1}{y}\frac{\partial Y}{\partial z} P + \left(\frac{\partial Y}{\partial y} - \frac{z}{y}\frac{\partial Y}{\partial z}\right) Q + Y = 0. \tag{19}$$

On can easily check using induction on k that a solution to (19) can be obtained in the form

$$Y(z, y) = y\left(1 + \sum_{k=1}^{\infty} g_k(z) y^k\right),$$

where $g_k(z)$ is a polynomial of degree k. Then, $Y\left(\frac{x}{y}, y\right)$ is a series in x and y of the form $Y(x, y) = y + \sum_{k+j=2}^{\infty} Y_{kj} x^k y^j$. Since

$$\Psi = \frac{xy\sqrt[4]{-2x - \frac{y}{2} + 1}}{\left(6x - \frac{3y}{2} + 1\right)^{3/4}}$$

is a first integral of (18), the first equation of (18) is linearizable by the substitution $X = \Psi/Y$. □

To obtain the necessary conditions for cases (β) and (γ), presented in theorems 3 and 4, respectively, we found the minimal associate primes of the corresponding ideals $I_6 = \langle i_1, j_1, \ldots, i_6, j_6\rangle$. For these ideals, we were able to complete all the calculations with Singular over the field of rational numbers.

Theorem 3. *System (6) with $a_{01} = 1, b_{10} = 0$ is linearizable at the origin if and only if $a_{11} = -a_{10}, b_{11} = 0$ and one of the following conditions holds:*

(1) $b_{20} = a_{02} - b_{01} + 2 = a_{10} = 0$,
(2) $b_{20} = b_{01} + b_{02} - 1 = a_{02} + b_{02} + 1 = 0$,
(3) $b_{20} = a_{20} = 0$,
(4) $b_{02} = b_{01} + 2 = a_{02} = 2a_{20} + b_{20} = a_{10} = 0$,
(5) $b_{02} + 8 = b_{01} - 6 = a_{02} = 2a_{20} - b_{20} = a_{10} = 0$,
(6) $b_{02}^2 + 10b_{02} - 4 = 2b_{01} - b_{02} - 20 = 2a_{02} + 3b_{02} + 24 = -5b_{20}b_{02} + 8a_{20} + 6b_{20} = a_{10} = 0$.

Proof. We now prove that they are the sufficient conditions. In case 1, system (6) with conditions (1) takes the form

$$\dot{x} = x(1 - a_{20}x^2 - y + (2 - b_{01})y^2), \qquad \dot{y} = -y(1 - b_{01}y - b_{02}y^2). \tag{20}$$

System (20) has two invariant lines given by

$$\ell_{1,2} = 1 + \frac{1}{2}\left(-b_{01} \pm \sqrt{b_{01}^2 + 4b_{02}}\right)y.$$

Moreover, we can construct an inverse integrating factor of the form $V = x^3y^3\ell_1^{\beta_1}\ell_2^{\beta_2}$, where

$$\beta_{1,2} = \frac{(b_{01} - 2)\left(b_{01} + b_{02} \pm \sqrt{b_{01}^2 + 4b_{02}}\right)}{b_{02}\sqrt{b_{01}^2 + 4b_{02}}}.$$

Multiplying both parts of (20) by $\ell_1^{\beta_1}\ell_2^{\beta_2}$, we obtain a system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{21}$$

such that $V = x^3y^3$ is an inverse integrating factor for (21), and the coefficients of x^3y^2 and x^2y^3 in the series expansions for P and Q, respectively, are equal to zero. Therefore, by theorem 4.13, and (4.28) of [11], system (20) admits a first integral Ψ of the form (3). (Note that using the integrating factor it is possible to find a first integral of (20) in the closed form, but it is a long expression involving the Appell hypergeometric function.)

Looking for a solution of the equation $Q(x, y)/y + \sum \alpha_i K_i = -1$, we find that the second equation is linearizable by the substitution $z_2 = y\ell_1^{\alpha_1}\ell_2^{\alpha_2}$, where $\alpha_{1,2}$ are given by

$$\alpha_{1,2} = -\frac{1}{2} \pm \frac{b_{01}}{2\sqrt{b_{01}^2 + 4b_{02}}}. \tag{22}$$

Finally, using the existence of the analytic first integral Ψ the first equation is linearizable by the substitution $z_1 = \Psi/z_2$.

In case 2, system (6) with conditions (2) takes the form

$$\dot{x} = x(1 - a_{10}x - a_{20}x^2 - y + a_{10}xy + (1 + b_{02})y^2), \qquad \dot{y} = -y(1 - (1 - b_{02})y - b_{02}y^2). \tag{23}$$

System (23) has four invariant lines given by

$$\ell_1 = 1 - y, \qquad \ell_2 = 1 + b_{20}y, \qquad \ell_{3,4} = 1 + \tfrac{1}{2}\left(-a_{10} \pm \sqrt{a_{10}^2 + 4a_{20}}\right)x - y.$$

Moreover, system (23) has a first integral of the form $\Phi = xyl_2^{\beta_2}l_3^{\beta_3}l_4^{\beta_4}$ with

$$\beta_2 = -\frac{1 + b_{02}}{b_{02}}, \qquad \beta_{3,4} = -\frac{1}{2} \pm \frac{a_{10}}{2\sqrt{a_{10}^2 + 4a_{20}}}, \tag{24}$$

and system (23) is linearizable by the substitutions

$$z_1 = xl_1^{\alpha_1}l_2^{\alpha_2}, \qquad z_2 = yl_1^{-\alpha_1}l_2^{\beta_2 - \alpha_2}l_3^{\alpha_3}l_4^{\alpha_4},$$

where $\alpha_1 = 1/(1 + b_{02})$, $\alpha_2 = (-1 - 2b_{02})/(b_{02}(1 + b_{02}))$ and $\alpha_{3,4} = \beta_{3,4}$ given in (24).

In case 3, system (6) with conditions (3) takes the form

$$\dot{x} = x(1 - a_{10}x - y + a_{10}xy - a_{02}y^2), \qquad \dot{y} = -y(1 - b_{01}y - b_{02}y^2). \tag{25}$$

This system admits the invariant lines

$$\ell_{1,2} = 1 + \tfrac{1}{2}\left(-b_{01} \pm \sqrt{b_{01}^2 + 4b_{02}}\right)y,$$

which allow us to construct an inverse integrating factor of the form $V = x^2y^2\ell_1^{\beta_1}\ell_2^{\beta_2}$, where

$$\beta_{1,2} = \frac{a_{02}\left(\pm b_{01} + \sqrt{b_{01}^2 + 4b_{02}}\right) + b_{02}\left(\mp 2 \pm b_{01} + \sqrt{b_{01}^2 + 4b_{02}}\right)}{2b_{02}\sqrt{b_{01}^2 + 4b_{02}}}.$$

Using the equation $Q(x, y)/y + \sum \alpha_i K_i = -1$, we see that the second equation is linearizable by the substitution $z_2 = y\ell_1^{\alpha_1}\ell_2^{\alpha_2}$, where $\alpha_{1,2}$ are given in (22). Using the existence of the inverse integrating factor V it is possible to find a first integral of (25) in the closed form (3), but it is a long expression involving the Appell hypergeometric function. On the other hand, multiplying both parts of (25) by $\ell_1^{\beta_1}\ell_2^{\beta_2}$ we obtain a system, such that $V = x^2y^2$ is an inverse integrating factor for it, and the coefficients of x^2y and xy^2 in the series expansions for the first and the second equations, respectively, are equal to zero. Thus by theorem 4.13 and (4.28) of [11] system (25) has a first integral Ψ of the form (3). Using the integral, the first equation is linearizable by the substitution $z_1 = \Psi/z_2$.

Case 4 is a particular case of the dual to case 3 under the involution $a_{ij} \leftrightarrow b_{ji}$.

In case 5, the system has two cubic invariant algebraic curves given by

$$\ell_1 = 1 - 12y + 12b_{20}x^2y + 48y^2 - 64y^3, \qquad \ell_2 = 1 - 4y + 4b_{20}x^2y.$$

Moreover, the system has a first integral of the form $\Psi = \ell_1^{-1/3}\ell_2$ and the expansion of $\Psi - 1$ has the form $\Psi - 1 = x^2y^2 + \cdots$. Applying theorem 1, the system is linearizable by the change of variables $z_1 = x^2y\ell_1^{-2/3}\ell_2(\Psi - 1)^{-1/2}$ and $z_2 = x^{-2}y^{-1}\ell_2(\Psi - 1)$.

In case 6, system (6) with conditions (6) takes the form

$$\begin{aligned} \dot{x} &= x(1 - (5b_{01}b_{20} - 53b_{20})x^2/4 - y - (18 - 3b_{01})y^2), \\ \dot{y} &= -y(1 - b_{20}x^2 - b_{01}y - (2b_{01} - 20)y^2), \end{aligned} \tag{26}$$

where $b_{01} = (15 \pm \sqrt{29})/2$. We have not been able to find more algebraic curves than $x = 0$ and $y = 0$. But applying theorem 1, system (26) is linearizable by the change of variables $z_1 = x^{(1-c)}y^{-c}\Psi^c r^a$ and $z_2 = \Psi/z_1$, where r and Ψ are an inverse integrating factor and a first integral of system (26), respectively, and

$$a = \frac{-956 \pm 463\sqrt{29}}{1495}, \qquad c = \frac{3(-853 \pm 349\sqrt{29})}{1495}.$$

To complete this case it remains to prove the existence of an analytic first integral Ψ and consequently the existence of the integrating factor r. We are going to prove that system (26) has an analytic first integral by induction. First, doing the map $\{z = y/x, x = x\}$ system (26) becomes

$$\begin{aligned} \dot{x} &= x[4 + (53 - 5b_{01})b_{20}x^2 - 4xz + (12b_{01} - 72)x^2z^2]/4, \\ \dot{z} &= -z[8 + (49 - 5b_{01})b_{20}x^2 - (4 + 4b_{01})xz + (8 + 4b_{01})x^2z^2]/4. \end{aligned} \tag{27}$$

System (27) admits an analytic first integral of the form

$$\psi(x, z) = x^2z\sum_{k=2}^{\infty} f_k(z)x^k,$$

and one can easily see that the functions $f_k(z)$ can be found recursively in the form $f_k(z) = zp_{k-1}(z)$ where $p_{k-1}(z)$ is a polynomial of degree $k-1$. The polynomials $f_k(z)$ satisfy the following recurrence differential equation:

$$a(z)f_{k-2} + b(z)f_{k-1} + c(z)f'_{k-2} + d(z)f'_{k-1} - 2z^2 f'_k(z) + (k-1)zf_k(z) = 0, \tag{28}$$

for $k \geqslant 4$, where $a(z) = (53k + 4 - 5kb_{01})\, b_{02}z/4 + ((3k + 2)b_{01} - 18k - 20)z^3$, $b(z) = (b_{01} - (k + 1))z^2$, $c(z) = (5b_{01} - 49)b_{20}z^2/4 - (2z + b_{01})z^4$, $d(z) = (1 + b_{01})z^3$. Calculations yield that we can choose $f_2(z) = z$ and $f_3(z) = 2(-1 + b01)z^2$. Assume that for $k = 4, \ldots, m$, there are polynomials $f_k(z)$ satisfying (28) and such that $\deg(f_k) = k$, then for $k = m + 1$, solving the linear differential equation (28), we obtain

$$f_{m+1}(z) = Cz^{\frac{m+2}{2}} + \frac{1}{2}z^{\frac{m+2}{2}} \int z^{-\frac{m+4}{2}} h_{m+2},$$

where $h_{m+2} = a(z)f_{m-1} + b(z)f_m + c(z)f'_{m-1} + d(z)f'_m$. Since h_{m+2} is a polynomial of degree $m + 2$, taking $C = 0$ it is easily seen that $\deg f_{m+1} = m + 1$. Thus, the inductional claim is fulfilled. Hence, going back to the original coordinates $\Psi = \sqrt{\psi(x, y/x)}$ is a first integral of the form (3). □

Remark. Some functions appearing in the proofs of theorems 1 and 2 are not defined for specific values of parameters. The existence of an integral (3) for these specific values follows from the fact that the set of all systems (6) with a center at the origin is a closed set in the Zariski topology.

Theorem 4. *System (6) with $a_{01} = b_{10} = 0$ is linearizable at the origin if and only if $a_{11} = b_{11} = 0$ and one of the following conditions holds:*

(1) $b_{20} = a_{02} = 0$,
(2) $b_{20} = a_{20} = 0$,
(3) $a_{02} + b_{02} = a_{20} + b_{20} = 0$,
(4) $b_{02} = a_{02} = 0$.

Proof. In case 1, the system has the invariant lines,

$$\ell_{1,2} = 1 + \tfrac{1}{2}\left(-b_{01} \pm \sqrt{b_{01}^2 + 4b_{02}}\right)y, \qquad \ell_{3,4} = 1 + \tfrac{1}{2}\left(-a_{10} \pm \sqrt{a_{10}^2 + 4a_{20}}\right)x,$$

and it is linearizable by the substitutions

$$z_1 = x\ell_3^{\alpha_3}\ell_4^{\alpha_4}, \qquad z_2 = y\ell_1^{\alpha_1}\ell_2^{\alpha_2},$$

where

$$\alpha_{1,2} = -\frac{1}{2} \pm \frac{b_{01}}{2\sqrt{b_{01}^2 + 4b_{02}}}, \qquad \alpha_{3,4} = -\frac{1}{2} \pm \frac{a_{10}}{2\sqrt{a_{10}^2 + 4a_{20}}}. \tag{29}$$

In case 2, system (6) with conditions (2) becomes

$$\dot{x} = x(1 - a_{10}x - a_{02}y^2), \qquad \dot{y} = -y(1 - b_{01}y - b_{02}y^2). \tag{30}$$

Similarly to case 3 of theorem 3, system (30) has only two invariant straight lines given by

$$\ell_{1,2} = 1 + \frac{1}{2}\left(-b_{01} \pm \sqrt{b_{01}^2 + 4b_{02}}\right)y.$$

However, we can construct an inverse integrating factor of the form $V = x^2y^2\ell_1^{\beta_1}\ell_2^{\beta_2}$ where

$$\beta_{1,2} = \frac{(a_{02} + b_{02})\left(\pm b_{01} + \sqrt{b_{01}^2 + 4b_{02}}\right)}{2b_{02}\sqrt{b_{01}^2 + 4b_{02}}}.$$

Using the equation $Q(x, y)/y + \sum \alpha_i K_i = -1$, we see that the second equation is linearizable by the substitution $z_2 = y\ell_1^{\alpha_1}\ell_2^{\alpha_2}$, where $\alpha_{1,2}$ are given in (29). Finally, using the existence of the an analytic first integral Ψ of the form (3) guaranteed by the same argument as for case 3 of theorem 3, the first equation is linearizable by the substitution $z_1 = \Psi/z_2$.

In case 3, system (6) with conditions (3) takes the form

$$\dot{x} = x(1 - a_{10}x + b_{20}x^2 + b_{02}y^2), \qquad \dot{y} = -y(1 - b_{01}y - b_{20}x^2 - b_{02}y^2). \tag{31}$$

System (31) has four invariant lines given by

$$\ell_{1,2} = 1 - \tfrac{1}{2}\left(a_{10} \pm \sqrt{a_{10}^2 + 4a_{20}}\right)x - \tfrac{1}{2}\left(b_{01} \pm \sqrt{b_{01}^2 + 4b_{02}}\right)y,$$
$$\ell_{3,4} = 1 - \tfrac{1}{2}\left(a_{10} \pm \sqrt{a_{10}^2 + 4a_{20}}\right)x - \tfrac{1}{2}\left(b_{01} \mp \sqrt{b_{01}^2 + 4b_{02}}\right)y.$$

Moreover, system (31) has a first integral of the form $\Psi = xy\ell_1^{-1}\ell_2^{-1}$. Applying theorem 1, system (31) is linearizable by the change of variables $z_1 = x\ell_1^{\alpha_1}\ell_2^{\alpha_3}\ell_3^{\alpha_5}$ and $z_2 = y\ell_1^{\alpha_2}\ell_2^{\alpha_4}\ell_3^{-\alpha_5}$, where $\alpha_{1,2,3,4}$ are given in (29) and $\alpha_5 = -\alpha_1 - \alpha_3 - 1$.

Case 4 is the dual to case 2 under the involution $a_{ij} \leftrightarrow b_{ji}$. □

4. Concluding remarks

We have presented the solution to the linearizability problem for the general complex cubic system of autonomous differential equations with two invariant lines. From our results we can easily obtain the classification of all isochronous centers in the cubic system (5) with two complex lines $u \pm \mathrm{i}v = 0$. To obtain the conditions for isochronicity of such a system from the conditions of linearizability of system (6) we can set

$$\begin{aligned} a_{10} &= a_1 + \mathrm{i}b_1, \qquad a_{01} = a_2 + \mathrm{i}b_2, \qquad a_{20} = a_3 + \mathrm{i}b_3, \\ a_{02} &= a_4 + \mathrm{i}b_4, \qquad a_{11} = -(a_2 + \mathrm{i}b_2)a_1 \end{aligned} \tag{32}$$

and consider the equation

$$\dot{x} = \mathrm{i}x(1 - a_{10}x - a_{01}\bar{x} - a_{20}x^2 - a_{11}x\bar{x} - a_{02}\bar{x}^2). \tag{33}$$

Substituting in this equation instead of a_{kj} expressions (32) and $x = u + \mathrm{i}v$, and then separating the real and imaginary parts of (33) we obtain from (33) a system

$$\dot{u} = -u + \tilde{P}_2(u, v) + \tilde{P}_3(u, v), \qquad \dot{v} = v + \tilde{Q}_2(u, v) + \tilde{Q}_3(u, v). \tag{34}$$

Then substituting expressions (32) with $a_2 = 1$, $b_2 = 0$ into the conditions (1–8) of theorem 2 we obtain nine conditions for isochronicity of system (34). Similarly, substituting expressions (32) with $a_2 = b_2 = 0$ into the conditions (1–4) of theorem 4 we obtain four more conditions for isochronicity of (34). Since every system (5) with the isochronous center at the origin and the invariant lines $u \pm \mathrm{i}v = 0$ can be transformed to system (34), and systems studied in theorem 3 have no real counterpart, the obtained 13 conditions present the solution of the isochronicity problem for system (5) with the invariant lines $u \pm \mathrm{i}v = 0$ (we do not write down the conditions here; however, the interested reader can easily compute them in the way described). In particular, performing this computation one obtains the answer to the first part of problem 1: 'what are the isochronous systems inside the family (A)'. In fact, inspecting the proof of theorems 2 and 4 we see that the isochronous systems inside (A) are the systems corresponding to cases (1), (2), (4), (5), (8) of theorem 2 and all cases of theorem 4 (in particular, system (3)–(5) of [32] corresponds to the case (1) of our theorem (2)). As to the second part of problem 1: 'what are the Darboux linearizable systems inside family (A)', we see that all systems mentioned above except for case (8) of theorem 2 are Darboux linearizable. We conjecture that the latter system is not Darboux linearizable in the sense of [32, 33]; however, at present we cannot prove this conjecture.

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